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Convergence Property of Relaxation Rabling Under Two-Way Sinkhon Normalization

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Abstract: This article describes the convergence property of relaxation Rabling under two-way Sinkhon normalization. This method can be used in non-rigid image registration and point matching. In this article, we have used Shinkon normalization to prove it.

Keywords: Point matching, Relaxation, Normalization, Medical imaging.

1. Introduction

The point matching is widely used in computer vision and pattern recognition because point representations are generally easy to extract. Point matching can be categorized as rigid matching and non-rigid matching based on the deformation of objects captured in the image. Compare with the rigid case, non-rigid matching is more complicated. Generally, there are two unknown operations: correspondence and transformation. Most non-rigid point matching approaches use an iterated estimation framework to find appropriate correspondence and transformation [1]. The Iterated Closest Point (ICP) algorithm is one of the most well known heuristic approaches. It utilizes the relationship by assigning the correspondence with binary values zero and one. However, this binary assumption is no longer valid in the case of non-rigid transformation, especially when the deformations are large [2]. The Thin Plate Spline Robust Point Matching (TPS-RPM) algorithm is an Expectation Maximization (EM) algorithm to jointly solve for the feature correspondence as well as the geometric transformation. The cost function that is being minimized is the sum of Euclidean distances between points. In the TPS-RPM, the binary correspondence value of the ICP is relaxed to the continuous value between zero and one. This soft-assign method improves the matching performance because the correspondences are able to improve gradually and continuously without jumping around in the space of binary permutation matrices [3-4]. The algorithm is robust compared to the ICP in the non-rigid case, but the joint estimation of correspondences and transformation increases complexity. In addition, the Euclidean distance makes sense only when there are at least rough initial alignments of the shapes. If the initialization is not aligned well, the matching result is poor. Recently, the Shape Context (SC) algorithm has been proposed. It is an object recognizer based on the shape. For each point, the distribution of the distance and orientation are estimated to the neighboring points through a histogram [5]. This distribution is used as the attribute relations for the points. The correspondences can be decided by comparing each point's attributes in one set with the attributes of the other. Because only the attributes are compared, the searching for the correspondences can be conducted more easily compared to the ICP and the TPS-RPM. Generally, the SC performs better in handling complex patterns than the TPS-RPM. Another interesting approach of point matching is a kernel correlation-based point matching. The cost function is proportional to the correlation of two

kernel density estimates. The work was extended by using the L2 distance between mixtures of Gaussian representing the point set data in. The Coherent Point Drift (CPD) algorithm is another probabilistic algorithm [6]. They proposed to use the displacement filed between the point sets following the motion coherence theory. They also successfully extend the general non-rigid registration framework, and show that TPS-RPM is its special case [7]. The Robust Point Matching by preserving Local Neighborhood Structures (RPM-LNS) introduces the notion of a neighborhood structure for the general point matching problem. The RPM-LNS uses a relaxation labeling method with binary value coefficient. This approach is based on the assumption that although the absolute distance between two points may change significantly under non-rigid deformation, the neighborhood structure of a point is generally well preserved. The cost function is formulated as an optimization problem to preserve local neighborhood relations. This research shows the convergence property of relaxation Rabling under two-way Sinkhon normalization. This method can be used in non-rigid image registration and point matching. In this paper, we used Shinkon normalization to prove it.

2. Problem Definition

Let $S=\{s_1,s_2,...,s_M\}$ be a set of points in a model shape and $T=\{t_1,t_2,...,t_N\}$ be a set of points in the target shape. In a point matching problem, one-to-one matching is desired, but in general, one-to-one matching is not possible because of outliers. To handle this problem, two point sets are augmented to $S'=\{s_1,s_2,...,s_M,nil\}$ and $T'=\{t_1,t_2,...,t_N,nil\}$ by introducing a dummy point *nil*. Then a match between shapes S and T is $f:S'\Leftrightarrow T'$ and common points can be matched one-to-one and outliers can be matched to a dummy point *nil*. Under a rigid transformation (transformation and rotation), the distance between any pair of points is preserved. Therefore, the optimal match $\hat{f}$ is

$$\hat{f}=arg\min_f C(S,T,f),$$

where

$$C(T,D,f)=\sum_{m=1}^{M}\sum_{i=1}^{M}(\|s_m-s_i\|-\|f(s_m)-f(s_i)\|)^2 +\sum_{n=1}^{N}\sum_{j=1}^{N}(\|t_n-t_j\|-\|f(t_n)-f(t_j)\|)^2.$$

If a non-rigid transformation is present, the distance between a pair of points will not be preserved, especially for points which are far apart. On the other hand, due to physical constraints, and in order to preserve the rough structure, the local neighborhood of a point may not change freely. We therefore define the local neighborhood of a point. For a given point, $s_m\in S'$, a neighbor point is $\mathcal{N}_i^{s_m}$, $i=1,2,...,I$. Similarly, for a given point, $t_n\in T'$, a neighbor point is $\mathcal{N}_j^{t_n}$, $j=1,2,...,J$. Since the only distance of neighboring point pairs are preserved, (2) becomes

$$C(T,D,f)=\sum_{m=1}^{M}\sum_{i=1}^{I}(\|s_m-\mathcal{N}_i^{s_m}\|-\|f(s_m)-f(\mathcal{N}_i^{s_m})\|)^2 +\sum_{n=1}^{N}\sum_{j=1}^{J}(\|t_n-\mathcal{N}_j^{t_n}\|-\|f(t_n)-f(\mathcal{N}_j^{t_n})\|)^2$$

We quantize the distance to two levels as

$$\|s_m-\mathcal{N}_i^{s_m}\|=\begin{cases}1 \text{ if } s_m\in\mathcal{N}_i^{s_m}\\ 0 \text{ if } s_m\notin\mathcal{N}_i^{s_m}\end{cases},$$

$$\|f(s_m)-f(\mathcal{N}_i^{s_m})\|=\begin{cases}1 \text{ if } f(s_m)\in\mathcal{N}_i^{f(s_m)}\\ 0 \text{ if } f(s_m)\notin\mathcal{N}_i^{f(s_m)}\end{cases},$$

$$\|t_n-\mathcal{N}_j^{t_n}\|=\begin{cases}1 \text{ if } t_n\in\mathcal{N}_j^{t_n}\\ 0 \text{ if } t_n\notin\mathcal{N}_j^{t_n}\end{cases},$$

$$\|f(t_n)-f(\mathcal{N}_j^{t_n})\|=\begin{cases}1 \text{ if } f(t_n)\in\mathcal{N}_j^{f(t_n)}\\ 0 \text{ if } f(t_n)\notin\mathcal{N}_j^{f(t_n)}\end{cases}.$$

Equation ( ) then is simplified to

$$C(T,D,f)=\sum_{m=1}^{M}\sum_{i=1}^{I}d(s_m,\mathcal{N}_i^{s_m}) +\sum_{n=1}^{N}\sum_{j=1}^{J}d(t_n,\mathcal{N}_j^{t_n}),$$

where

$$d(s_m,\mathbb{N}_i^{s_m})\in\begin{cases}1 \text{ if } s_m\notin\mathbb{N}_i^{s_m} \text{ or } f(s_m)\notin\mathbb{N}_i^{f(s_m)}\\ 0 \text{ if } s_m\in\mathbb{N}_i^{s_m} \text{ and } f(s_m)\in\mathbb{N}_i^{f(s_m)}\end{cases}$$

and

$$d(t_n,\mathcal{N}_j^{t_n})\in\begin{cases}1 \text{ if } t_m\notin\mathcal{N}_j^{t_m} \text{ or } f(t_m)\notin\mathcal{N}_j^{f(t_m)}\\ 0 \text{ if } t_m\in\mathcal{N}_j^{t_m} \text{ and } f(t_m)\in\mathcal{N}_j^{f(t_m)}\end{cases}.$$

Simple deduction makes to convert the above minimization problem to a maximization problem.

$$\hat{f}=arg\max_f K(S,T,f),$$

$$K(T,D,f)=\sum_{m=1}^{M}\sum_{i=1}^{I}\delta(s_m,\mathcal{N}_i^{s_m}) + \sum_{n=1}^{N}\sum_{j=1}^{J}\delta(t_n,\mathcal{N}_j^{t_n}) \ ,$$

where $\delta(i,j)=1-d(i,j).$

In this paper, the point matching problem is restated as the graph isomorphism problem [12]. The optimal solution of ( ) is the one that maximizes the number of matched edges of two graphs. Each point is a node of a graph, and a point and its adjacent point constitute the edges of the graph. Then the problem is to maximize the number of matched edges between two graphs. For this purpose, we determine the fuzzy correspondence matrix P with dimension $(M+1)\times(N+1)$. Each entry of P has continuous value between [0,1] that indicates the weight of the correspondence between s_m and t_n. The optimal match $\hat{P}$ is found by maximizing the energy function as follows.

$$\hat{P}=arg\max_{P}E(P),$$

where

$$E(P)=\sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{n=1}^{N}\sum_{j=1}^{J}P_{mn}P_{ij}.$$

subject to

$$\sum_{m=1}^{M+1}P_{mn}=1,\ \forall n,\ \sum_{n=1}^{N+1}P_{mn}=1,\ \forall m,\ P_{mn}\in[0,1].$$

3. Relaxation Labeling

Start with a set of nodes i and a set of labels λ. Derive a set of compatibility coefficients r for each problem of interest and then apply the basic recipe of relaxation labeling for updating the node-label assignments:

$$q_i(\lambda)=\sum_{j=1}\sum_{\mu=1}r_{ij}(\lambda,\mu)p_j(\mu),$$

$$p_i(\lambda)^{(n+1)}=\frac{p_i(\lambda)^{(n)}(1+\alpha q_i(\lambda)^{(n)})}{\sum_{\mu=1}p_i(\mu)^{(n)}(1+\alpha q_i(\mu)^{(n)})}$$

Now, we prove that the relaxation labeling process always converges with two way constraints ( ). The common framework of prediction algorithm is as follows.

$$F(p,\sigma)=E(p)+\frac{1}{\alpha}d(p,\sigma),$$

where $d(p,\sigma)$ is a distance measure between p and an "old" value σ.

Examine the following objective function using the generalized KL divergence as the distance measure:

$$\begin{aligned}E(M,\sigma,\mu,\nu,\alpha)=&-\sum_{i=1}^{I}\sum_{b=1}^{B}\sum_{j=1}^{J}\sum_{a=1}^{A}M_{p_iq_j}M_{\mathcal{N}_a^{p_i}\mathcal{N}_b^{q_j}}\\&+\frac{1}{\alpha}\sum_{i=1}^{I}\sum_{j=1}^{J}(M_{p_iq_j}\log\frac{M_{p_iq_j}}{\sigma_{p_iq_j}}\\&-M_{p_iq_j}+\sigma_{p_iq_j})\\&+\sum_{i=1}^{I+1}\mu_i(\sum_{j=1}^{J+1}M_{p_iq_j}-1)\\&+\sum_{j=1}^{J+1}\nu_j(\sum_{i=1}^{I+1}M_{p_iq_j}-1)\end{aligned}$$

Instead of solving for the Lagrange parameter vectors μ and ν, we use Sinkorn's theorem instead to ensure that the row and column constrains satisfied. From our assumption of exact convergence of Sinkhorn, it follows that the Lagrange parameter vectors μ and ν can be dropped from the energy function.

We first formalize a few definitions.

Algorithm 1: (Shinkhorn Scaling Algorithm)

Given a nonnegative, $m\times n$ matrix A, and specified vectors of the row sums ($r\in\mathbb{R}^m$) and column sums ($c\in\mathbb{R}^m$), we iterate the following until convergence, with initial values $a_{ij}^{(0)}=a_{ij}$, and $k=1$:

1) Multiple every element $a_{ij}^{(k-1)}$ by the ratio of the desired row sum r_i to the actual row sum

$$\sum_{j=1}^{n}a_{ij}^{(k-1)},\ a_{ij}^{(k)}=r_ia_{ij}^{(k-1)}/(\sum_{j=1}^{n}a_{ij}^{(k-1)})$$

2) Multiple every element $a_{ij}^{(k)}$ of the matrix from (1) by the ratio of the desired column sum c_j to the actual column sum $\sum_{i=1}^{m}a_{ij}^{(k)}$, $a_{ij}^{(k^*)}=c_ja_{ij}^{(k)}/(\sum_{i=1}^{m}a_{ij}^{(k)})$.

It can be shown that with any given matrix A, Sinkorn scaling process will converge to a unique matrix B that satisties the row and column sum constraints. The following theorem is a unified statement of the convergence of the Sinkorn scaling process, from various previous results in the literature.

Theorem 1: Consider $A\in\mathbb{R}^{m\times n}$, a nonnegative matrix, and desired row sums $r\in\mathbb{R}^m$ and column sums $c\in\mathbb{R}^n$. Then there exists a unique matrix

$B \in \mathbb{R}^{m \times n}$ which satisties these prescribed row and column sums, where $B = D_1 A D_2$ for $D_1 \in \mathbb{R}^{m \times m}$ and $D_2 \in \mathbb{R}^{n \times n}$, D_1 and D_2 both diagonal, positive definite matrices.

If we assume that Sinkhorn procedure always returns a doubly stochastic matrix.

Then we have used the generalized KL divergence which is guaranteed to be greater than or equal to zero without requiring the usual constraints

$$\Delta F \overset{def}{=} F(M^{(t)}) - F(M^{(t+1)})$$
$$= -\sum_{i=1}^{I}\sum_{b=1}^{B}\sum_{j=1}^{J}\sum_{a=1}^{A} M_{p_i q_j}^{(t)} M_{\mathcal{N}_a^{p_i}\mathcal{N}_b^{q_j}}^{(t)}$$
$$+\frac{1}{\alpha}\sum_{i=1}^{I}\sum_{j=1}^{J}(M_{p_i q_j}^{(t)} \log\frac{M_{p_i q_j}^{(t)}}{\sigma_{p_i q_j}} - M_{p_i q_j}^{(t)} + \sigma_{p_i q_j})$$
$$+\sum_{i=1}^{I}\sum_{b=1}^{B}\sum_{j=1}^{J}\sum_{a=1}^{A} M_{p_i q_j}^{(t+1)} M_{\mathcal{N}_a^{p_i}\mathcal{N}_b^{q_j}}^{(t+1)}$$
$$-\frac{1}{\alpha}\sum_{i=1}^{I}\sum_{j=1}^{J}(M_{p_i q_j}^{(t+1)} \log\frac{M_{p_i q_j}^{(t+1)}}{\sigma_{p_i q_j}}$$
$$-M_{p_i q_j}^{(t+1)} + \sigma_{p_i q_j})$$

$$E(M,\sigma,\mu,\nu,\alpha) = -\sum_{i=1}^{I}\sum_{b=1}^{B}\sum_{j=1}^{J}\sum_{a=1}^{A} M_{p_i q_j} M_{\mathcal{N}_a^{p_i}\mathcal{N}_b^{q_j}}$$
$$+\frac{1}{\alpha}\sum_{i=1}^{I}\sum_{j=1}^{J}(M_{p_i q_j} \log\frac{M_{p_i q_j}}{\sigma_{p_i q_j}} - M_{p_i q_j} + \sigma_{p_i q_j})$$
$$+\sum_{i=1}^{I+1}\mu_i(\sum_{j=1}^{J+1} M_{p_i q_j} - 1)$$
$$+\sum_{j=1}^{J+1}\nu_j(\sum_{i=1}^{I+1} M_{p_i q_j} - 1)$$

$$\Delta F \overset{def}{=} \sum_{i=1}^{I}\sum_{b=1}^{B}\sum_{j=1}^{J}\sum_{a=1}^{A} \Delta M_{p_i q_j} \Delta M_{\mathcal{N}_a^{p_i}\mathcal{N}_b^{q_j}}$$
$$+\frac{1}{\alpha}\sum_{i=1}^{I}\sum_{j=1}^{J} M_{p_i q_j}^{(t)} \log\frac{M_{p_i q_j}^{(t)}}{M_{p_i q_j}^{(t+1)}} \geq 0$$

$$\Delta F \overset{def}{=} F(M^{(t)}) - F(M^{(t+1)})$$
$$= -\sum_{i=1}^{I}\sum_{b=1}^{B}\sum_{j=1}^{J}\sum_{a=1}^{A} M_{p_i q_j}^{(t)} M_{\mathcal{N}_a^{p_i}\mathcal{N}_b^{q_j}}^{(t)}$$
$$+\frac{1}{\alpha}\sum_{i=1}^{I}\sum_{j=1}^{J}(M_{p_i q_j}^{(t)} \log\frac{M_{p_i q_j}^{(t)}}{\sigma_{p_i q_j}} - M_{p_i q_j}^{(t)} + \sigma_{p_i q_j})$$
$$+\sum_{i=1}^{I}\sum_{b=1}^{B}\sum_{j=1}^{J}\sum_{a=1}^{A} M_{p_i q_j}^{(t+1)} M_{\mathcal{N}_a^{p_i}\mathcal{N}_b^{q_j}}^{(t+1)}$$
$$-\frac{1}{\alpha}\sum_{i=1}^{I}\sum_{j=1}^{J}(M_{p_i q_j}^{(t+1)} \log\frac{M_{p_i q_j}^{(t+1)}}{\sigma_{p_i q_j}} - M_{p_i q_j}^{(t+1)} + \sigma_{p_i q_j}),$$

where $\Delta M_{p_i q_j} \overset{def}{=} \Delta M_{p_i q_j}^{(t+1)} - \Delta M_{p_i q_j}^{(t)}$ and $\Delta M_{\mathcal{N}_a^{p_i}\mathcal{N}_b^{q_j}} \overset{def}{=} \Delta M_{\mathcal{N}_a^{p_i}\mathcal{N}_b^{q_j}}^{(t+1)} - \Delta M_{\mathcal{N}_a^{p_i}\mathcal{N}_b^{q_j}}^{(t)}$. The first term is nonnegative due to the positive definiteness in the subspace spanned by the row and column constraints. The second term is non-negative by virtue of being a KL distance measure. We have shown the convergence to a fixed point of the relaxation labeling under Sinkhorn scaling.

This is a consequence of the following extension to the well-known Birkoff-von Neumann theorem: the set of doubly stochastic matrices is the convex hull of the set of permutation matrices and outliers. So it can be ensured that we will always achieve one-to-one correspondence.

Complexity and Convergence of Sinkhorn Scaling

The Sinkhorn iterations are a natural way of scaling a matrix to achieve prescribed row and column sums. While Sinkorn proved that the iterative procedure converges for appropriate matrices, it could take a very long time to reach a desired accuracy ($b_{ij} < \varepsilon$). In our algorithm, however, we set $b_{ij} < 0.95$ as an outlier by matching them a dummy point and set as 0. This way improves the rate of convergence of Sinkorn process significantly.

4. Transformation Function

Given a finite set of correspondences between, one can proceed to estimate a plane transformation. $f: \Re^2 \to \Re^2$ or $f: \Re^3 \to \Re^3$ that may be used to map arbitrary points from one image to the other. In this study, we mostly use the thin plate spline (TPS) model, which is commonly used for representing flexible coordinate transformations. Bookstein found it to be highly effective for modeling changes in biological forms. Powell applied the TPS model to recover transformations between curves. The thin plate spline is the 2D generalization of the cubic spline. In its regularized form, which is discussed below, the TPS model includes the affine model as a special case.

Let v_i denote the target function values at corresponding locations $p_i = (x_i, y_i)$ in the plane, with $i = 1, 2, ..., n$. In particular, we will set v_i equal to x_i' and y_i' in turn to obtain one continuous transformation for each coordinate. We assume that the locations (x_i, y_i) are all different and are not collinear. In 2-D interpolation problem, the TPS interpolant $f(x, y)$ minimizes the bending energy

$$I_f = \int\int_{\mathbb{R}^2}\left[\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 f}{\partial y^2}\right)^2\right] dxdy$$

and has the form

$$f(x,y)=a_1+a_x x+a_y y+\sum_{i=1}^{n} w_i U(\|(x_i,y_i)-(x,y)\|)$$

If the problem is 3-D interpolation, the bending energy is

$$I_f=\iiint_{\mathbb{R}^3}\left[\begin{array}{l}\left(\frac{\partial^2 f}{\partial x^2}\right)^2+\left(\frac{\partial^2 f}{\partial y^2}\right)^2+\left(\frac{\partial^2 f}{\partial z^2}\right)^2\\ +2\left(\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 f}{\partial x\partial z}\right)^2+\left(\frac{\partial^2 f}{\partial y\partial z}\right)^2\right)\end{array}\right]dxdydz$$

and the interpolant form is

$$f(x,y,z)=a_1+a_x x+a_y y+a_z z+\sum_{i=1}^{n} w_i U(\|(x_i,y_i,z_i)-(x,y,z)\|)$$

The kernel function $U(r)$ is defined by $U(r)=r^2\log r^2$ and $U(0)=0$ as usual. In order for $f(x,y)$ to have square integrable second derivatives, we require the boundary condition as

$$\sum_{i=1}^{n} w_i=0 \text{ and } \sum_{i=1}^{n} w_i x_i=\sum_{i=1}^{n} w_i y_i=0.$$

A special characteristic of the thin-plate spline is that the resulting transformation is always decomposed into a global transformation and a local non-affine warping component. The first three terms in 2-D and four terms in 3-D case describes global affine transform and rest terms describe non-linear (nonglobal) transformation.

Together with the interpolation conditions, $f(x_i,y_i)=v_i$, this yields a linear system for the TPS coefficients:

$$\begin{pmatrix}K & P\\ P^T & 0\end{pmatrix}\begin{pmatrix}W\\ A\end{pmatrix}=\begin{pmatrix}V\\ 0\end{pmatrix},$$

where $K=\begin{bmatrix}0 & U(r_{12}) & \cdots & U(r_{1n})\\ U(r_{21}) & 0 & \cdots & U(r_{2n})\\ \cdots & \cdots & \cdots & \cdots\\ U(r_{n1}) & U(r_{n2}) & \cdots & 0\end{bmatrix}$

and $P=\begin{bmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ \cdots & \cdots & \cdots\\ 1 & x_n & y_n\end{bmatrix}$.

Here, $r_{ij}=\|P_i-P_j\|$ is the distance between points P_i and P_j. W and A is column vectors formed from $W=(w_1,w_2,...,w_n)$ and $A=(a_1,a_x,a_y)$, respectively. $V=(v_1,v_2,...,v_n)$ is any n-vector. We will denote the $(n+3)\times(n+3)$ matrix $\begin{pmatrix}K & P\\ P^{\mathrm{T}} & 0\end{pmatrix}$ by L. Since L is nonsingular, we can find the solution by inverting L [Powell 1995]. Define the vector $Y=(V\,|\,0\;\;0\;\;0)^{\mathrm{T}}$, then $(W\,|\,a_1\;\;a_x\;\;a_y)^{\mathrm{T}}=L^{-1}Y$. If we denote the upper left $n\times n$ block of L^{-1} by L_p^{-1}, then it can be shown that $I_f\propto v^T L_p^{-1}v=w^T Kw$.

When there is noise in the specified value v_i, one may wish to relax the exact interpolation requirement by means of regularization. This is accomplished by minimizing

$$H[f]=\sum_{i=1}^{n}(v_i-f(x_i,y_i))^2+\lambda I_f$$

The regularization parameter λ, a positive scalar, controls the amount of smoothing; the limiting case of $\lambda=0$ reduces to exact interpolation. As demonstrated in [ ], we can solve for the TPS coefficients in the regularized case by replacing the matrix K by $K+\lambda I$.

In the application we take the points (x_i,y_i) to be landmarks and V to be $n\times 2$ matrix, $V=\begin{bmatrix}x_1' & x_2' & \cdots & x_n'\\ y_1' & y_2' & \cdots & y_n'\end{bmatrix}^{\mathrm{T}}$ Y to be $(n+3)\times 2$ matrix, $Y=\begin{bmatrix}x_1' & x_2' & \cdots & x_n' & | & 0 & 0 & 0\\ y_1' & y_2' & \cdots & y_n' & | & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$ where each (x_i',y_i') is the control points homologous to (x_i,y_i) in another copy of $\Re^2$. The application of L^{-1} to the first column of V^{T} specifies the coefficient of 1, x, y, and the U's for $f_x(x,y)$, the x - coordinate of the image of (x,y). The application of L^{-1} to the second column of V^{T} does the same for the y - coordinate $f_x(x,y)$. The resulting function $f(x,y)=[f_x(x,y),f_y(x,y)]$ is nor vector-valued and it maps each point (x_i,y_i) to its homolog (x_i',y_i') and is least bent of all such functions. These vector valued functions $f(x,y)$ are the thin-plate spline mappings.

5. 3-D Point Context

We treat an object as a point set and we assume that the shape of an object is essentially captured by a finite subset of its points. For each point on the first shape, we want to find the best matching point. We propose a 3-D novel descriptor, the 3-D point context, which could play such a role in shape matching. Consider the set of vectors originating from a point to all other sample points on a shape. These vectors express the configuration of the entire shape relative to the reference point.

The full set of vectors as a shape descriptor is much too detailed since shapes and their sampled representation may vary of one instance to another in a category. We identify the distribution over relative points as a more robust and compact, yet highly discriminative descriptor. For a point p_i on the shape, we compute a coarse histogram h_i of the relative coordinates of the remaining $n-1$ points, $h_i(k) = \#\{q \neq p_i : (q - p_i) \in \text{bin}(k)\}$.

The histogram is defined to be the shape context of p_i. We use bins that are uniform in log-polar space, making the descriptor more sensitive to positions of nearby sample points than to those of points farther away.

The log-polar geometry was first motivated by its resemblance with the structure of the retina of some biological vision systems and by its data compression qualities. The log-polar transformation is a conformal mapping from the points on the Cartesian plane (x, y) to points in the log-polar plane (ξ, η).

The mapping is described by

$$\xi = \log\sqrt{x^2 + y^2}$$
$$\eta = a\tan(y / x)$$

Consider a point p_i on the first shape and a point q_j on the second shape. Let $C_{ij} = C(p_i, q_j)$ denote the cost of matching these two points. As shape contexts are distributions represented as histograms, it is natural to use the chi-square test statistics:

$$C_{ij} \equiv C(p_i, q_j) = \frac{1}{2}\sum_{k=1}^{K}\frac{[h_i(k) - h_j(k)]^2}{h_i(k) + h_j(k)},$$

where $h_i(k)$ and $h_j(k)$ denote the K-bin normalized histogram at p_i and q_j, respectively.

6. The Background of Nystrom Approximation

The computation cost of TPS becomes prohibitive when the number of samples is large. Let n be the number of samples of a deformation map. TPS require the solution of a $n \times n$ dense system with $O(n^3)$ complexity for determining interpolation coefficients.

One drawback of the TPS model is that its solution requires the inversion of a large dense matrix of size $n \times n$, where n is the number of points in the data set. In this section, the approximation method that addresses this computational problem is discussed.

Since inverting L is an $O(n+3)^3$ operation, solving for the TPS coefficients can be very expensive when n is very large. We will now discuss the Nystrom approximation method that reduces this computational burdon. The Nystrom method is a technique for finding numerical approximations to eigenfunction problems of the form:

$$\int_a^b K(x, y)\phi(y)dy = \lambda\phi(x)$$

We can approximate this integral equation by evaluating it as a set of evenly spaced points $\xi_1, \xi_2, ..., \xi_n$ on the interval $[a, b]$ and employing a simple quadrature rule,

$$\frac{(b-a)}{n}\sum_{j=1}^{n} K(x, \xi_j)\hat{\phi}(\xi_j) = \lambda\hat{\phi}(x),$$

where $\hat{\phi}(x)$ is the approximation to the true $\phi(x)$. To solve the above, we set $x = \zeta_i$ yielding the system of equations

$$\frac{(b-a)}{n}\sum_{j=1}^{n} K(\xi_i, \xi_j)\hat{\phi}(\xi_j) = \lambda\hat{\phi}(\xi_i) \quad \forall i \in \{1, 2, ..., n\}$$

Without loss of generality, we let $[a, b]$ be $[0,1]$ and the structure the system as the matrix eigenvalue problem:

$$K\hat{\Phi} = n\hat{\Phi}\Lambda,$$

where $K_{ij} = k(y_i, y_j)$ is the Gram matrix and $\Phi = [\phi_1\ \phi_2\ ...\ \phi_n]$ are n approximate eigenvectors with corresponding eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Substituting back into equation yields Nystrom extension for each $\hat{\phi}_i$

$$\hat{\phi}_i(x) = \frac{1}{n\lambda_i}\sum_{j=1}^{n} K(x, \zeta_j)\hat{\phi}_i(\zeta_j)$$

7. Approximating the Eigenvectors of Affinity Matrices

The preceding analysis suggests that it should be possible to find approximate eigenvectors of a large Gram matrix by solving a much smaller eigenproblem using only a subset of the entries and employing the Nystrom extension to fill in the rest.

Consider a Gram matrix $K \in \mathbb{R}^{p \times p}$ partitioned as follows $K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{m \times m}$, where $p = n + m$ and we will take n to be much smaller than m. Since K is positive definite, we can write it as the inner product of a matrix Z with itself: $K = Z^T Z$. If K is of rank n and the rows of the submatrix $[A, B]$ are linearly independent, Z can be written using only A and B as follows. Let Z be

partitioned $Z=[X\ Y]$ with $X\in\mathbb{R}^{p\times n}$ and $Y\in\mathbb{R}^{p\times m}$. Rewriting K we have:

$$K=Z^TZ=\begin{bmatrix}X^TX & X^TY\\ Y^TX & Y^TY\end{bmatrix}$$

Then $A=X^TX$ and $B=X^TY$. Using the diagonalization $A=U\Lambda U^T$, where $U^TU=I$ we obtain

$$X=\Lambda^{1/2}U^T$$
$$\hat{Y}=(X^T)^{-1}B=(\Lambda^{1/2}U^T)^{-1}B=\Lambda^{-1/2}U^TB$$

Combining the two into $\hat{Z}=[X\ \hat{Y}]\in\mathbb{R}^{P\times P}$ gives us

$$\begin{aligned}K&=\begin{bmatrix}X^TX & X^T\hat{Y}\\ \hat{Y}^TX & \hat{Y}^T\hat{Y}\end{bmatrix}\\&=\begin{bmatrix}X^TX & (\Lambda^{1/2}U^T)^T\Lambda^{-1/2}U^TB\\ (\Lambda^{-1/2}U^TB)^T\Lambda^{1/2}U^T & (\Lambda^{-1/2}U^TB)^T\Lambda^{-1/2}U^TB\end{bmatrix}\\&=\begin{bmatrix}A & B\\ B^T & B^TA^{-1}B\end{bmatrix}\\&=\begin{bmatrix}A\\ B^T\end{bmatrix}A^{-1}[A\ B]\end{aligned}$$

Again $A=U\Lambda U^T$ be its eigendecomposition where U has orthonormal columns and Λ is diagonal. Letting $\tilde{U}$ denote the approximate eigenvectors of K, the Nystrom extension gives

$$\tilde{U}=\begin{bmatrix}U\\ B^TU\Lambda^{-1}\end{bmatrix}$$

And the associated approximation of K, which we denote $\hat{K}$, then takes the form

$$\begin{aligned}\hat{K}=\tilde{U}\Lambda\tilde{U}^T&=\begin{bmatrix}U\\ B^TU\Lambda^{-1}\end{bmatrix}\Lambda[U^T\ \ \Lambda^{-1}U^TB^T]\\&=\begin{bmatrix}U\Lambda U^T & B\\ B^T & B^T\Lambda B\end{bmatrix}=\begin{bmatrix}A & B\\ B^T & B^T\Lambda B\end{bmatrix}\\&=\begin{bmatrix}A\\ B^T\end{bmatrix}A^{-1}[A\ B]\end{aligned}$$

Note that in general the columns of $\tilde{U}$ are not orthogonal.

This is addressed as follows. If A is positive definite, then we can solve the orthogonalized approximate eigenvectors. Let $A^{1/2}$ denote the symmetric positive definite square root of A, define $S=A+A^{-1/2}BB^TA^{-1/2}$ and diagonalize it as $S=U_S\Lambda_SU_S^T$. If the matrix V is defined as

$$V=\begin{bmatrix}A\\ B^T\end{bmatrix}A^{-1/2}U_S\Lambda_S^{-1/2}$$

Then one can show that $\hat{W}$ is diagonalized by V and Λ_S, i.e. $\hat{K}=V\Lambda_SV^T$ and $V^TV=I$.

$$\begin{aligned}\hat{K}&=\{\begin{bmatrix}A\\ B^T\end{bmatrix}A^{-1/2}U_S\Lambda_S^{-1/2}\}\Lambda\{\Lambda_S^{-1/2}U_SA^{-1/2}[A\ B]\}\\&=V\Lambda V^T\end{aligned}$$

$$\begin{aligned}I&=V^TV\\&=\{\Lambda_S^{-1/2}U_S^TA^{-1/2}[A\ B]\}\{\begin{bmatrix}A\\ B^T\end{bmatrix}A^{-1/2}U_S\Lambda_S^{-1/2}\}\end{aligned}$$

By multiplying from the left by

$$\begin{aligned}U_S\Lambda_SU_S^T&=A^{-1/2}[A\ B]\begin{bmatrix}A\\ B^T\end{bmatrix}A^{-1/2}\\&=A+A^{-1/2}BB^TA^{-1/2}=S\end{aligned}$$

From the standard formula for the partitioned inverse of L, we have

$$\begin{aligned}L^{-1}&=\begin{bmatrix}K & P\\ P^T & 0\end{bmatrix}^{-1}\\&=\begin{bmatrix}(K^{-1}+K^{-1}PQ^{-1}P^TK^{-1} & -K^{-1}PQ^{-1}\\ -Q^{-1}P^TK^{-1} & Q^{-1}\end{bmatrix}\end{aligned}$$

with $Q=-P^TK^{-1}P$.

Thus

$$\begin{aligned}\begin{bmatrix}W\\ A\end{bmatrix}&=\begin{bmatrix}K & P\\ P^T & 0\end{bmatrix}^{-1}\begin{bmatrix}V\\ 0\end{bmatrix}\\&=\begin{bmatrix}(K^{-1}+K^{-1}PQ^{-1}P^TK^{-1} & -K^{-1}PQ^{-1}\\ -Q^{-1}P^TK^{-1} & Q^{-1}\end{bmatrix}\begin{bmatrix}V\\ 0\end{bmatrix}\\&=\begin{bmatrix}(K^{-1}+K^{-1}PQ^{-1}P^TK^{-1})V\\ -Q^{-1}P^TK^{-1}V\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix}W\\ A\end{bmatrix}&=\begin{bmatrix}K & P\\ P^T & 0\end{bmatrix}^{-1}\begin{bmatrix}V\\ 0\end{bmatrix}\\&=\begin{bmatrix}(K^{-1}+K^{-1}PQ^{-1}P^TK^{-1} & -K^{-1}PQ^{-1}\\ -Q^{-1}P^TK^{-1} & Q^{-1}\end{bmatrix}\begin{bmatrix}V\\ 0\end{bmatrix}\\&=\begin{bmatrix}(K^{-1}+K^{-1}PQ^{-1}P^TK^{-1})V\\ -Q^{-1}P^TK^{-1}V\end{bmatrix}\end{aligned}$$

Using the Nystrom approximation to K, we have $\hat{K}=V\Lambda_SV^T$ and $\hat{K}^{-1}=V\Lambda_S^{-1}V^T$

$$\hat{W} = (I + \hat{K}^{-1} P \hat{Q}^{-1} P^T) \hat{K}^{-1} V$$
$$= (I + V \Lambda_S^{-1} V^T P \hat{Q}^{-1} P^T) V \Lambda_S^{-1} V^T V$$
$$\hat{A} = -\hat{Q}^{-1} P^T \hat{K}^{-1} V = -\hat{Q}^{-1} P^T V \Lambda_S^{-1} V^T V$$

with $\hat{Q} = -P^T \hat{K}^{-1} P = -P^T V \Lambda_S^{-1} V^T P$ with is 3×3. Therefore, by computing matrix vector products in the appropriate order, we can obtain estimates to the TPS coefficients without having to invert or store a large $(n+3) \times (n+3)$ matrix. For the regularized case, one can proceed in the same manner, using

$$(\hat{K} + \lambda I)^{-1} = (V \Lambda_S V^T + \lambda I)^{-1} = V (\Lambda_S + \lambda I)^{-1} V^T$$

Finally, the approximate bending energy is given by

$$I_f = w^T \hat{K} w = w^T V \Lambda_S V^T w = (V^T w)^T \Lambda_S (V^T w)$$

Note that this bending energy is the average of the energies associated to the x and y components.

7. Conclusions

This research shows the convergence property of relaxation Rabling under two-way Sinkhon normalization. This method can be used in non-rigid image registration and point matching. In this paper, we used Shinkon normalization to prove it.

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